ON THE SOLUTION OF THE TURBULENCE PROBLEM BY THE METHOD OF PERTURBATION THEORY

(O RESHENII PROBLEMY TURBULENTNOSTI METODOM TEORII VOZMUSHCHENII)

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We shall describe the statistical properties of the turbulent motions of an incompressible fluid in an unbounded space in a random force field $\mathbf{X}(\mathbf{x}, t)$ with the aid of the joint characteristic functional of the velocity field $\mathbf{u}(\mathbf{x}, t)$ and the random force field

$$\Omega\,[\boldsymbol{\theta}(\mathbf{x}, t), \mathbf{f}(\mathbf{x}, t)] = \langle \exp\{i(\boldsymbol{\theta}\cdot\mathbf{u}) + i(\mathbf{f}\cdot\mathbf{X})\}\rangle \qquad (1)$$

where the parentheses indicate integration of the scalar product of the functions over all $\mathbf{x}$ and all t, and the angular brackets indicate the mean value operation. This functional gives a complete statistical description of the velocity and external force fields in the sense that for functions of the form

$$\boldsymbol{\theta}(\mathbf{x}, t) = \sum_{n=1}^{N} \boldsymbol{\theta}_n \delta(\mathbf{x} - \mathbf{x}_n)\,\delta(t - t_n), \qquad \mathbf{f}(\mathbf{x}, t) = \sum_{n=1}^{N} \mathbf{f}_n \delta(\mathbf{x} - \mathbf{x}_n)\,\delta(t - t_n) \qquad (2)$$

its values are the characteristic functions of the probability distributions for the values of the fields under consideration for all finite point sets $(\mathbf{x}_n, t_n)$ of space-time. Let us consider the characteristic functional for the function set $\boldsymbol{\theta}(\mathbf{x}, t)$ and $\mathbf{f}(\mathbf{x}, t)$, which permit Fourier transformation with respect to $\mathbf{x}$, and let us turn to its wave representation, setting

$$\Lambda\,[\mathbf{z}(\mathbf{k}, t), \mathbf{g}(\mathbf{k}, t)] = \Omega\left[(2\pi)^{-3}\int e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{z}(\mathbf{k}, t)\,d\mathbf{k}, (2\pi)^{-3}\int e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{g}(\mathbf{k}, t)\,d\mathbf{k}\right] \qquad (3)$$

In the work of Hopf [1] it is shown that the characteristic functional of the velocity field satisfies a certain linear equation in the variational derivatives which arises from the Navier-Stokes equations and the continuity equation. An analogous equation also holds for the functional Λ. It has the form

$$\left[\left(\frac{\partial}{\partial t} + \nu k^2\right) D_{z_j}(\mathbf{k}, t) - \Delta_{j\beta}(\mathbf{k})\, D_{g_\beta}(\mathbf{k}; t)\right]\Lambda =$$

$$= \Delta_{j\beta}(\mathbf{k})\, k_\alpha \int d\mathbf{k}' D_{z_\alpha}(\mathbf{k}', t)\, D_{z_\beta}(\mathbf{k} - \mathbf{k}', t)\, \Lambda \qquad \left(\Delta_{jl}(\mathbf{k}) = \delta_{jl} - \frac{k_j k_l}{k^2}\right) \tag{4}$$

(an equation of very similar form is derived in the work of Kraichman and Lewis [2]). Here ν is the corfficient of viscosity, $D_{z_j}(\mathbf{k}, t)$ and $D_{g_j}(\mathbf{k}, t)$ are variational differentiation operators * with respect to $z_j(\mathbf{k}, t)$ and $g_j(\mathbf{k}, t)$, and summation is implied by repeating Greek indices. We note that the left-hand side of Equation (4) comes from the linear terms of the Navier-Stokes equations and the right-hand side from the nonlinear terms.

The condition of solenoidality of the velocity field (expressed by the continuity equation for an incompressible fluid) reduced to a functional Λ [1] that must depend only on a component of the vector function $\mathbf{z}(\mathbf{k},t)$ which is orthogonal to the wave vector $\mathbf{k}$. Let us denote such a function by $\mathbf{z}^*(\mathbf{k}, t)$, so that $z_j^*(\mathbf{k}, t) = \Delta_{j\alpha}(\mathbf{k})\, z_\alpha(\mathbf{k}, t)$. Analogously, if the external force field is solenoidal, Λ will then depend only on a corponent of $\mathbf{g}^*(\mathbf{k},t)$ of the function $\mathbf{g}(\mathbf{k},t)$.

Let us consider a random external force field as given and being satistically stationary in time. Obviously, a statistically stationary velocity field will correspond to it, and the characteristic functional Λ of these two fields will be a solution of Equation (4) satisfying condition

$$\Lambda\,[0;\, \mathbf{g}\,(\mathbf{k}, t)] = G\,[\mathbf{g}\,(\mathbf{k}, t)] \tag{5}$$

where G is the given characteristic functional of the external force field.

The determination of such a solution will permit a complete statistical description of stationary turbulence to be given. Random force fields are, generally speaking, fictitious (and in problems where they are real as, for example, the Archimedian forces for thermal convection, they must not be prescribed, but must be determined together with the hydrodynamic fields), but let us prescribe them so that an energy influx is provided on the average only to the large-scale components of turbulence. Then it is to be expected that the fictitious character of the external force will not affect the statistical characteristics of the small-scale components of the turbulence, and the latter will be correctly described by the solution of Equation (4) for condition (5).

As the external force field let us choose a solinoidal, Gaussian, station-

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* Formula

$$\delta F = \int d\mathbf{k}\, dt\, \delta a_\beta(\mathbf{k}, t)\, D_{a_\beta}(\mathbf{k}, t)\, F$$

for the principal linear part δF of the variation of the functional F for an infinitessimal variation $\delta\mathbf{a}(\mathbf{k},t)$ of its functional argument serves to define the variational derivative $D_{a_j}(\mathbf{k}, t)\, F$ of the functional $F[\mathbf{a}(\mathbf{k},t)]$ with the function a_j at the point $(\mathbf{k},t)$.

ary, homogeneous and isotropic random field with zero average value. Its characteristic functional will have the form

$$G\ [\mathbf{g}\ (\mathbf{k}, t)] = \exp\left\{-\frac{1}{2}\int g_{\alpha}^{*}(\mathbf{k}, t_1)\ g_{\alpha}^{*}(-\mathbf{k}, t_2)\, f(k, |t_1 - t_2|)\, d\mathbf{k}\, dt_1\, dt_2\right\} \quad (6)$$

The function $f(k, \tau)$ which completely determines this functional is equivalent to the spatial Fourier transform of the space-time correlation function of the external forces. More precisely,

$$\langle X_{\alpha}\ (\mathbf{x}_1, t_1)\ X_{\beta}\ (\mathbf{x}_2, t_2)\rangle = \int e^{i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}\Delta_{\alpha\beta}\ (\mathbf{k})\ f\ (k, |t_2 - t_1|)\ d\mathbf{k} \quad (7)$$

Let us solve Equation (4) by the method of perturbation theory. Namely, let us note that the fluid which is in motion represents a system with internal interactions which for the Eulerian description of the motion will be inertial interactions between the spatial inhomogeneities of the velocity field described by the nonlinear terms in the Navier-Stokes equations (including the pressure gradient which is expressed by quadratic sums of the velocities at the same moment of time). Moreover, the ratio of typical values of the inertial forces to the viscous forces, i.e. the Reynolds number R, will be a constant of the interaction. The method of perturbation theory consists in regarding the effect of the interactions as a disturbance imposed on a linear system and seeking a solution of the nonlinear dynamical equations in the form of a series in powers of the constant of interaction. Thus, we will seek a formal solution of Equation (4) in the form of a series in powers of the Reynolds number. It can be expexted that the series will be convergent for small values of R, i.e. it will represent an exact solution (which, it is true, will be useful in describing only weak turbulence). But for large values of R the formal solution of Equation (4) in the form of a series in powers of R can also be useful for a variety of purposes, for example, as a standard with which other approximate solutions can be compared.

The nonlinear terms of the Navier-Stokes equations are one order larger with respect to R than any of the linear terms. Therefore, the right-hand side of Equation (4) is one order larger with respect to R than any of the terms on the left-hand side. We shall take this circumstance into account to obtain equations for the successive terms of the expansion of the functional Λ in powers of R. In particular, the equation for the zero term Λ_0 is obtained from (4) by rejecting the right-hand side. As is not difficult to verify, the solution of such an equation for condition (5) has the form

$$\Lambda_0 = G\ [\boldsymbol{\zeta}^*\ (\mathbf{k}, t) + \mathbf{g}^*\ (\mathbf{k}, t)], \qquad \boldsymbol{\zeta}^*\ (\mathbf{k}, t) = \int_t^{\infty} e^{-\nu k^2 (\tau - t)}\, \mathbf{z}^*\ (\mathbf{k}, \tau)\ d\tau \quad (8)$$

Setting $\mathbf{g}^*\ (\mathbf{k}, t) \equiv 0$, here, we obtain the zero approximation for the characteristic functional of the velocity field. According to (6), the velocity field in this approximation will be Gaussian; its correlation function is found to be equal to

$$\langle u_\alpha(\mathbf{x}_1, t_1)\, u_\beta(\mathbf{x}_2, t_2)\rangle = \int e^{i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}\, d\mathbf{k}\, \frac{\Delta_{\alpha\beta}(\mathbf{k})}{2\nu k^2} \int_{-\infty}^{\infty} f(k, |\tau|)\, e^{-\nu k^2 |\tau+t_2-t_1|}\, d\tau \quad (9)$$

Hence, it follows that the kinetic energy spectrum of the turbulence in the zero*th* approximation with respect to the Reynolds number has the form

$$E_0(k) = \frac{2\pi}{\nu} \int_{-\infty}^{\infty} f(k, |\tau|)\, e^{-\nu k^2 |\tau|}\, d\tau \quad (10)$$

Let us write the expansion of the functional Λ in a series in powers of R in the form

$$\Lambda = \Lambda_0 \sum_{n=0}^{\infty} M_n\, [\boldsymbol{\zeta}^*(\mathbf{k}, t), \mathbf{g}^*(\mathbf{k}, t)] \quad (11)$$

Here M_n is a term of order R^n (where $M_0 \equiv 1$). Substituting this series into Equation (4), changing from variational differentiation with respect to $z_j(\mathbf{k},t)$ and $g_j(\mathbf{k},t)$ to differentiation with respect to $\zeta_j^*(\mathbf{k},t)$ and $g_j^*(\mathbf{k},t)$, making use of the definitions (8) and (6) of the functional Λ_0 and collecting terms of equal powers in R, we obtain an equation of the form

$$L_j(\mathbf{k}\ t)\, M_n = \{A_j^{(1)}(\mathbf{k}, t) + A_j^{(2)}[\boldsymbol{\xi}(\mathbf{k}, t); \mathbf{k}, t] + A_j^{(3)}[\boldsymbol{\xi}(\mathbf{k}, t); \mathbf{k}, t]\}\, M_{n-1} \quad (12)$$

$$(\boldsymbol{\xi}(\mathbf{k}, t) = \boldsymbol{\zeta}^*(\mathbf{k}, t) + \mathbf{g}^*(\mathbf{k}, t))$$

for the functionals M_n for $n \geqslant 1$.

Here L_j is a differential operator of first order defined by Formula

$$L_j(\mathbf{k}, t) = \Delta_{j\alpha}(\mathbf{k})\, [D_{\zeta_\alpha^*}(\mathbf{k}, t) - D_{g_\alpha^*}(\mathbf{k}, t)] \quad (13)$$

and $A_j^{(1)}$ is an operator containing repeated variational differentiation and defined by Formula (14)

$$A_j^{(1)}(\mathbf{k}, t) = \int d\mathbf{k}_1\, d\mathbf{k}_2\, dt_1\, dt_2\, a_{j,pq}(\mathbf{k}, t \mid \mathbf{k}_1, t_1 \mid \mathbf{k}_2, t_2)\, D_{\zeta_p^*}(\mathbf{k}_1, t_1)\, D_{\zeta_q^*}(\mathbf{k}_2, t_2)$$

$$a_{j,pq}(\mathbf{k}, t \mid \mathbf{k}_1, t_1 \mid \mathbf{k}_2, t_2) = \Delta_{j\alpha}(\mathbf{k})\, k_\beta \Delta_{\beta p}(\mathbf{k}_1)\, \Delta_{\alpha q}(\mathbf{k}_2)\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \times$$

$$\times\, e^{-\nu k_1^2 (t-t_1) - \nu k_2^2 (t-t_2)}\, \Phi(t - t_1)\, \Phi(t - t_2)$$

where $\Phi(t)$ is a function equal to 1 for $t > 0$, $\frac{1}{2}$ for $t = 0$ and 0 for $t < 0$; the operator $A_j^{(2)}$ is obtained from $A_j^{(1)}$ by replacing the operator $D_{\zeta_p^*}(\mathbf{k}_1, t_1) D_{\zeta_q^*}(\mathbf{k}_2, t_2)$ by the operator

$$-\int f(k_2, |t_2 - \tau|)\, \xi_q(-\mathbf{k}_2, \tau)\, d\tau D_{\zeta_p^*}(\mathbf{k}_1, t_1) -$$

$$-\int f(k_1, |t_1 - \tau|)\, \xi_p(-\mathbf{k}_1, \tau)\, d\tau D_{\zeta_q^*}(\mathbf{k}_2, t_2)$$

lastly, $A_j^{(3)}$ is a quadratic functional obtained from $A_j^{(1)}$ by replacing both operators $D_{\zeta_j^*}(\mathbf{k}, t)$ by the functions

$$\int f(k, |t - \tau|)\, \xi_j(-\mathbf{k}, \tau)\, d\tau$$

The functionals $M_1, M_2, \ldots$ are determined consecutively from Equations (12), for which each time a solution of the inhomogeneous equation

$$L_j(\mathbf{k}, t)\, M_n = F_j^{(n)}$$

with a known (determined from the previous step) right-hand side must be sought which satisfies condition

$$M_n\,[0, \mathbf{g}^*(\mathbf{k}, t)] = 0 \tag{15}$$

(condition (15) for $n \geqslant 1$ is a consequence of condition (5)). Let us denote such a solution by the symbol

$$M_n = L_j^{-1}(\mathbf{k}, t)\, F_j^{(n)}$$

We will then have

$$M_n = (S_1 + S_2 + S_3)\, M_{n-1}, \qquad S_r = L_j^{-1} A_j^{(r)} \quad (r = 1, 2, 3) \tag{16}$$

Thus, for example, it is not difficult to be convinced that

$$M_1 = S_3 1 = \int \zeta_j^*(\mathbf{k}', t')\, A_j^{(3)}\,[\xi(\mathbf{k}, t); \mathbf{k}', t']\, d\mathbf{k}'\, dt' \tag{17}$$

This quantity is a homogeneous power functional of third power relative to the functions $\zeta^*(\mathbf{k},t)$ and $g^*(\mathbf{k},t)$. From the definition of the operators $A_j^{(r)}$ and S_r it follows that if M is a homogeneous power functional of power m, then the quantities S_1M, S_2M and S_3M are, respectively, homogeneous power functionals of powers $m-1$, $m+1$ and $m+3$ (with the exceptions that the operator S_1 vanishes for power functionals containing not more than one component of $\zeta_j^*(\mathbf{k},t)$ and S_2 vanishes for functionals which are independent of $\zeta^*(\mathbf{k},t)$). Making use of these rules and Formula (16), we are convinced that M_2 is a sum of homogeneous power functionals of 2, 4 and 6 powers, M_3 of 1, 3, 5, 7 and 9 powers, and in general

$$M_{2n} = \sum_{m=1}^{3n} M_{2n}^{(2m)}, \quad M_{2n+1} = \sum_{m=0}^{3n+1} M_{2n+1}^{(2m+1)} \tag{18}$$

where $M_n^{(m)}$ are homogeneous power functionals of power m.

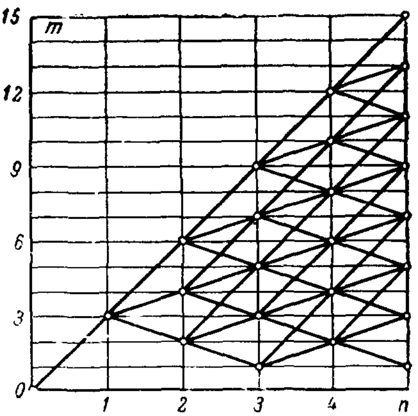

Fig. 1

The structure of the series (11) is shown on Fig.1, where the number n of the functionals M_n (which indicates the order with respect to R of the corresponding terms of the series) is plotted along the abscissa and the power m of the homogeneous power functionals $M_n^{(m)}$ along the ordinate. Moving along the graph from left to right, the lines directed downwards represent the operator S_1, those sloping gently upwards represent S_2 and those sloping steeply upwards represent S_3.

The graph provides information not only about the set of functionals $M_n^{(m)}$, which form M_n, but also about the method of calculating any functional $M_n^{(m)}$: for this it is necessary to sum the contributions corresponding to all possible polygonal paths with abscissa vertices not greater than n which terminate at the point (n,m). We shall designate such polygonal lines as diagrams which correspond to the functional $M_n^{(m)}$. Thus, for example, to the functional $M_2^{(2)}$ there corresponds one diagram (Fig.2) which represents the operator S_1S_3, so that $M_2^{(2)} = S_1S_3 1$. It is not difficult to verify that this functional can be written in the form

$$M_2^{(2)} = \int d\mathbf{k}d\mathbf{k}'dtdt'\zeta_\beta^*(\mathbf{k},t)\left\{A_\beta^{(1)}(\mathbf{k},t)\,\zeta_\alpha^*(\mathbf{k}',t') - \frac{1}{2}\zeta_\alpha^*(\mathbf{k}',t')\,A_\beta^{(1)}(\mathbf{k},t)\right\}A_\alpha^{(3)}[\xi(\mathbf{k},t);\mathbf{k}',t'] \quad (19)$$

To the functional $M_4^{(2)}$ there corresponds three diagrams (Fig.3) which represent the operators $S_1^2S_2S_3$, $S_1S_2S_1S_3$ and $S_2S_1^2S_3$, so that

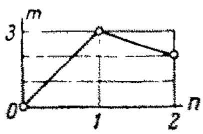

Fig. 2

$$M_4^{(2)} = (S_1^2S_2S_3 + S_1S_2S_1S_3 + S_2S_1^2S_3)\,1 \quad (20)$$

All of these expressions can be represented by one summary diagram (Fig.4). To the functional $M_6^{(2)}$ there already correspond 12 diagrams, etc.

The construction of diagrams of one or another form generally proves to be a useful way of representing a perturbation theory series. The most important example of this are the Feinman diagrams in quantum electrodynamics. Diagrams for the solution of the Navier-Stokes equations were constructed by Wyld [3], who also shoed how to construct diagrams for computing the correlation function of the velocity field from them.

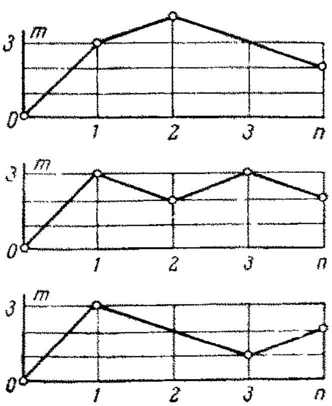

Fig. 3

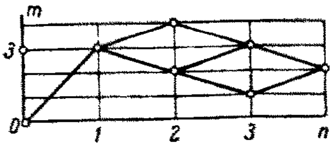

Fig. 4

With the help of variational differentiation of series (11) expansions in powers of R can be obtained for various statistical moments of the velocity field. Thus, for example, starting from the obvious formula

$$\langle u_\alpha(\mathbf{x}_1,t)\,u_\beta(\mathbf{x}_2,t)\rangle = -\int e^{-i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}d\mathbf{k}_1d\mathbf{k}_2[D_{z_\alpha}(\mathbf{k}_1,t)D_{z_\beta}(\mathbf{k}_2,t)\Lambda]_{z=g\equiv 0} \quad (21)$$

it is possible to obtain the expansion

$$E(k) = E_0(k) + S(k) \sum_{n=1}^{\infty} M_{2n}^{(2)} \tag{22}$$

for the kinetic energy spectrum $E(k)$, where $E_0(k)$ is given by Formula (10) and $S(k)$ is an operator which contains repeated variational differentiation and is defined by Formulas

$$S(k) = -\frac{k^2}{2}\int d\Omega(\mathbf{k})\, d\mathbf{k}_1 dt dt_1 a_{pq}(\mathbf{k}, t/\mathbf{k}_1, t_1)\, D\zeta_p^*(\mathbf{k}, t)\, D\zeta_q^*(\mathbf{k}_1, t_1)$$

$$a_{pq}(\mathbf{k}, t/\mathbf{k}_1, t_1) = \Delta_{\alpha p}(\mathbf{k})\, \Delta_{\alpha q}(\mathbf{k}_1)\, e^{\nu k^2 t + \nu k_1^2 t_1}\, \Phi(-t)\, \Phi(-t_1) \tag{23}$$

Here $d\Omega(\mathbf{k})$ is surface element of a unit sphere in wave vector space $(\mathbf{k})$.

The following two remarks can be made about the structure of the series (22): firstly, the expansion of the function $E(k)$ in a series in powers of the Reynolds number contains only even powers of R; secondly, it will simultaneously be an expansion of the function $E(k)$ in a functional power series relative to the function $f(k, |\tau|)$ (the external force spectrum) where the term of order R^{2n} (i.e. the term $S(k)M_{2n}^{(2)}$) is a homogeneous power functional of power $n+1$ relative to the function $f(k, |\tau|)$. The first of these remarks is trivial and the second follows from the diagrams of Fig.1 if it is taken into account that the operator S_1 does not change the power of the power functionals relative to $f(k, |\tau|)$, that S_2 raises the power by 1 and that S_3 raises the power by 2.

From the second observation it follows that the functional $S(k)M_{2n}^{(2)}$ is analogous in some degree to the "convolution" of $n+1$ functions f (i.e. to the probability density of the sum of $n+1$ independent random quantities, each of which has a probability density proportional to f). These "convolutions" have the following property: if f is different from zero only in the region of very small wave numbers $0 \leqslant k \leqslant k_0$, then in the region of very large wave numbers $k \gg k_0$ "convolutions" only of a very large number of functions f will be different from zero; but such "convolutions" will be only slightly dependent on the specific form of the function f (and will be close to some universal function of k corresponding to one or another of the so-called infinitely decomposable probability distribution laws). Analogously can be expected that if the spatial variation of the external forces has only large-scale inhomogeneities, i.e. if f is different from zero only for $0 \leqslant k \leqslant k_0$, then for $k \gg k_0$ the functionals $S(k)M_{2n}^{(2)}$ will be different from zero only for very large numbers $2n$, and these functionals will be only slightly dependent on the specific form of the function f, so that if the series (22) converges, then its sum $E(k)$ for $k \gg k_0$ will be close to some universal function of k. Thus, there arises the possibility of reaching an analytical proof of the hypothesis of A.N. Kolmogorov regarding a universal ststistical equilibrium of the small-scale components of turbulence.

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